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# A free-streamline solution for stratified flow into a line sink

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An analysis is made of the two-dimensional flow under gravity of an inviscid non-diffusive stratified fluid into a line sink, involving a velocity discontinuity in the flow field. The fluid above the discontinuity is stagnant and hence is not drawn into the sink. At sufficiently low values of the modified Froude number, this is the only physically possible mode of flow, and is the cause of flow separation in many industrial and natural processes. A proper mathematical solution for flows with a stagnant zone has so far been lacking. This paper presents such a solution, after posing the problem as one involving a free-streamline, which is the line of velocity discontinuity. The solution to be given here is obtained by an inverse method. It is also found herein that the modified Froude number has a value of 0·345 for all separated flows of the kind in question.

## 1. Introduction

We consider two-dimensional flow of an inviscid density-stratified fluid into a line sink in the bottom corner of a channel as shown in figure 1. For values of the modified Froude number $F$ (as defined below in equation (14$a$)) greater

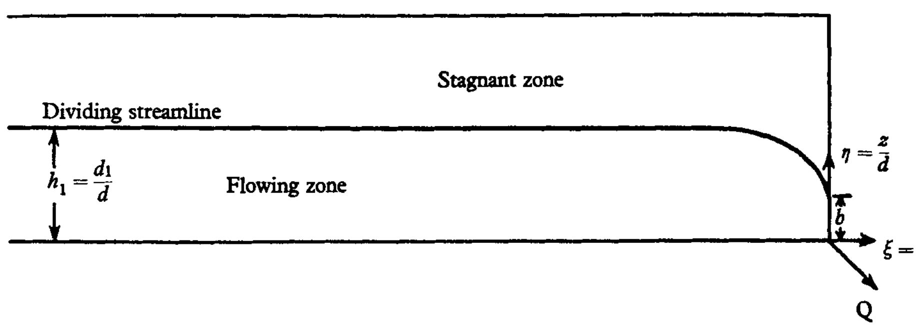


FIGURE 1. Flow into a sink.

than $\pi^{-1}$, a solution has been given by Yih (1958). However, this solution ceases to be valid in the neighbourhood of and below this value of $F$. It has been demonstrated experimentally by Debler (1959) that, when $F$ is near this critical value, the flow is characterized by the presence of a stagnant layer which is separated

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from the flow region by a line of velocity discontinuity. The proper mathematical solution for flow with a stagnant layer has to provide for a velocity discontinuity along a streamline in the flow field. This solution has so far been lacking and is presented in this paper. In the problem under consideration, the fluid on one side of the discontinuity is required to be stagnant, so that the pressure on the dividing streamline is known, although the position of this line is not known *a priori*. Bernoulli's equation is satisfied along the dividing streamline, and this provides a non-linear dynamic boundary condition on the flowing part. The solution to be given here is obtained by an inverse method and is exact in the sense that no approximating linearization or perturbation procedure is utilized to obtain the solution.

## 2. The governing equations

For steady two-dimensional flow of an inviscid, incompressible, density-stratified fluid in a gravitational field, with the gravity force acting in the negative $z$-direction, the equations of motion are

$$u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{1}$$

$$u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}-g, \tag{2}$$

in which $x$ and $z$ are Cartesian co-ordinates, $u$ and $w$ are the corresponding velocity components, $\rho$ is the density, $p$ is the pressure, and $g$ the gravitational acceleration. Assuming no diffusion, the density is constant along a streamline, or

$$u\frac{\partial \rho}{\partial x}+w\frac{\partial \rho}{\partial z} = 0, \tag{3}$$

and the equation of continuity for an incompressible fluid is

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z} = 0. \tag{4}$$

Instead of introducing directly a stream function $\psi$ at this stage, we can cast the above system of equations into a more convenient form by introducing an associated flow field (indicated by a prime) through the following transformation due to Yih (1958):

$$(u', w') = \sqrt{\frac{\rho}{\rho_0}}(u, w), \tag{5}$$

where $\rho_0$ is a reference density. With this transformation, and using (3), equations (1), (2) and (4) become

$$u'\frac{\partial u'}{\partial x}+w'\frac{\partial u'}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}, \tag{6}$$

$$u'\frac{\partial w'}{\partial x}+w'\frac{\partial w'}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z}-\frac{\rho}{\rho_0}g, \tag{7}$$

$$\frac{\partial u'}{\partial x}+\frac{\partial w'}{\partial z} = 0. \tag{8}$$

From equation (8), it follows that there exists a stream function for the associated flow, $\psi'$, such that

$$u' = -\frac{\partial\psi'}{\partial z}, \quad w' = \frac{\partial\psi'}{\partial x}.$$

From equation (3) and the transformation (5), it is obvious that density is still constant along a streamline in the associated flow field. Hence $\rho = \rho(\psi')$. Therefore, integration of (6) and (7) along a streamline shows that Bernoulli's equation is still valid along a streamline.

The equations of motion, (6) and (7), can now be combined into one equation governing $\psi'$,

$$\rho_0 \nabla^2\psi' = \frac{dH}{d\psi'} - gz\frac{d\rho}{d\psi'},$$

where $$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \quad \text{and} \quad H = \left(p + \frac{\rho_0(u'^2+w'^2)}{2} + g\rho z\right)$$

is the Bernoulli sum and is a function of $\psi'$ only. Writing $F(\psi')$ for $dH/d\psi'$, we have

$$\rho_0 \nabla^2\psi' + gz\frac{d\rho}{d\psi'} = F(\psi'). \tag{9}$$

This equation was originally obtained by Yih (1958). It possesses a form which is more suitable for further studies than the equation governing the stream function $\psi$ of the actual flow field, first obtained by Long (1953),

$$\nabla^2\psi + \frac{1}{\rho}\frac{d\rho}{d\psi}\left[\frac{1}{2}\left\{\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2\right\} + gz\right] = H(\psi). \tag{10}$$

It is easy to show that Long's equation can be simplified to (9) by Yih's transformation written in the form

$$\psi' = \int \left(\frac{\rho}{\rho_0}\right)^{\frac{1}{2}} d\psi.$$

It has been shown by Yih (1958) that, if the fluid originates from a large reservoir, where the velocity is zero and flows into the channel horizontally, the associated flow is irrotational far upstream. If we restrict our attention to a linear density stratification far upstream, equation (9) can be rendered exactly linear if $u'$ is a positive constant $A$ far upstream. For then, far upstream,

$$\psi' = -Az, \tag{11}$$

and if also the density stratification is linear,

$$\rho(-\infty, z) = \rho_0(1-\beta z), \quad \beta = (\rho_0 - \rho_1)/\rho_0 d,$$

where $\rho_0$ is the density at bottom of channel, $\rho_1$ the density at top of channel, and $d$ is the total depth of the channel. Then, using (11), (9) becomes

$$\nabla^2\psi' + k\psi' = -Akz, \tag{12}$$

where $k = g\beta/A^2$.

Equation (12) can be made dimensionless by defining

$$\Psi = \psi'/Ad, \quad \xi = x/d, \quad \eta = z/d. \tag{13}$$

The dimensionless form of (12) is then

$$\frac{\partial^2\Psi}{\partial\xi^2}+\frac{\partial^2\Psi}{\partial\eta^2}+F^{-2}\Psi = -F^{-2}\eta, \tag{14}$$

where

$$F = A(g\beta d^2)^{-\frac{1}{2}} \tag{14a}$$

is the modified Froude number. Equation (14) is the equation to be solved, subject to the boundary conditions to be considered in the next section.

## 3. The boundary conditions

With $\xi$ measured horizontally from the sink and $\eta$ measured vertically from the horizontal bottom of the channel, the boundary conditions are

$$\Psi = 0 \quad \text{for} \quad \xi < 0, \quad \eta = 0, \tag{15}$$

$$\Psi = -\eta \quad \text{for} \quad \xi = -\infty, \quad 0 \leqslant \eta \leqslant h_1, \tag{16}$$

$$\Psi = -h_1 \quad \text{for} \quad \xi = 0, \quad 0 < \eta < b, \tag{17}$$

where $h_1 = d_1/d$ is the dimensionless depth of the flow region far upstream, and $b$ is the point where the dividing streamline meets the line $\xi = 0$ (figure 1). Along the dividing streamline, $\Psi = -h_1$, Bernoulli's equation must be satisfied, which in non-dimensional form is

$$\frac{1}{2}\left\{\left(\frac{\partial\Psi}{\partial\xi}\right)^2+\left(\frac{\partial\Psi}{\partial\eta}\right)^2\right\}+\Pi+\left(\frac{\rho_s gd}{\rho_0 A^2}\right)\eta = \text{const.}, \tag{18}$$

where $\Pi = p/\rho_0 A^2$ is the dimensionless pressure, which is given by the hydrostatic pressure of the stable stagnant fluid, and $\rho_s$ is the density along the streamline.

From (18) some conditions regarding the velocity along the dividing streamline for a stable flow configuration can be derived. Physically these conditions are equivalent to saying that the density of the stagnant fluid must never increase upwards, and that in the neighbourhood of the streamline its density must be less than or equal to the density along the streamline. Thus, differentiation of (18) with respect to a dimensionless distance $s$ measured along the streamline yields

$$\frac{d}{ds}\left(\frac{q'^2}{2}\right) = -\left(\frac{\rho_s gd}{\rho_0 A^2}+\frac{d\Pi}{d\eta}\right)\frac{d\eta}{ds},$$

where $q'^2 \equiv (\partial\Psi/\partial\xi)^2+(\partial\Psi/\partial\eta)^2$. But, since the pressure distribution in the stagnant zone is hydrostatic,

$$dp = -\rho' g\, dz,$$

where $\rho'$ is the density in the stagnant zone, or

$$d\Pi = -(\rho' gd/\rho_0 A^2)\, d\eta,$$

so that

$$\frac{d}{ds}\left(\frac{q'^2}{2}\right) = -\left\{\frac{(\rho_s-\rho')\, gd}{\rho_0 A^2}\right\}\frac{d\eta}{ds}, \tag{19}$$

and

$$\frac{d}{d\eta}\left(\frac{q'^2}{2}\right) = -\left\{\frac{(\rho_s-\rho')\, gd}{\rho_0 A^2}\right\},$$

which upon further differentiation with respect to $\eta$ yields

$$\frac{d^2}{d\eta^2}\left(\frac{q'^2}{2}\right) = \frac{gd}{\rho_0 A^2}\frac{d\rho'}{d\eta}. \tag{20}$$

Now $d\eta/ds$ is negative in the geometry considered here and $d\rho'/d\eta$ is negative for a statically stable stratification. Thus, from (19) and (20), it follows that, for a stable flow configuration, the square of the velocity must be monotonically increasing and $d^2(\frac{1}{2}q'^2)/d\eta^2$ must be negative or zero along the dividing streamline. It then follows that the dividing streamline must meet the line $\xi = 0$ tangentially, for otherwise a stagnation point will result, making the flow unstable.

For the inverse method of solution of this investigation, the above conditions are utilized to produce *a posteriori* a stable stratification, as will be seen in the next section. It is to be noted that only the density of the fluid in the stagnant wedge between the dividing streamline and the horizontal tangent to the dividing streamline far upstream is of importance here. The density of the stagnant fluid above this region does not affect the problem.

In order that the solution may represent a physically realistic situation, the density of the stagnant wedge must be a constant. This is because, at the initiation of the flow, the layer of fluid above the dividing streamline is required to shift slightly to fill the wedge region. Furthermore, if the density profile far upstream is to be preserved, the density in the stagnant wedge must not only be constant but also equal the density along the dividing streamline. For a stagnant zone with constant density $\rho_A$, equation (18) gives

$$\frac{1}{2}\left\{\left(\frac{\partial\Psi}{\partial\xi}\right)^2 + \left(\frac{\partial\Psi}{\partial\eta}\right)^2 - 1\right\} = -\left(\frac{\rho_s-\rho_A}{\rho_0}\right)\frac{\beta^{-1}}{d}F^{-2}(\eta-h_1). \tag{21}$$

Therefore, when $\rho_s = \rho_A$

$$\left(\frac{\partial\Psi}{\partial\xi}\right)^2 + \left(\frac{\partial\Psi}{\partial\eta}\right)^2 = 1. \tag{22}$$

From this equation, the equation of motion and the kinematic boundary conditions, and with $d = d_1$, it is clear that $F$ is a constant for all separated flows and that the velocity profiles are similar, a result which has been noted by Yih (1964). This can be more easily seen if one observes that all the flow patterns are determined for one value of $F$ by virtue of the fact that the non-dimensionalizing is based on the depth of the discharging layer.

## 4. Method of solution

An inverse method will be used to solve the system given by equations (14) to (18). The method is to introduce a distribution of sinks $g(\eta)$ on $\xi = 0$. In this way the flow field is still continuous everywhere, but there is one streamline which divides the flow into two regions, one part flowing completely into the original sink and the other into the sink distribution that has been introduced. This new problem can be stated as follows:

$$\frac{\partial^2\Psi}{\partial\xi^2} + \frac{\partial^2\Psi}{\partial\eta^2} + F^{-2}\Psi = -F^{-2}\eta, \tag{23}$$

in which $\Psi$ is assumed to be of the form

$$\Psi = r\Psi_1 + (1-r)\Psi_2, \tag{24}$$

where $r$ represents the percentage of the total flow field that flows into the original sink, and $\Psi_1$ represents the flow into the original sink, satisfying the following boundary conditions:

$$\Psi_1 = -1 \quad \text{for} \quad \begin{Bmatrix} \xi = 0, & 0 < \eta \leqslant 1, \\ -\infty < \xi < 0, & \eta = 1, \end{Bmatrix} \tag{25}$$

$$\Psi_1 = 0 \quad \text{for} \quad -\infty < \xi < 0, \quad \eta = 0, \tag{26}$$

$$\Psi_1 = -\eta \quad \text{for} \quad \xi = -\infty, \quad 0 \leqslant \eta \leqslant 1; \tag{27}$$

$\Psi_2$ represents the flow into the sink distribution, satisfying the following boundary conditions:

$$\Psi_2 = -1 \quad \text{for} \quad -\infty < \xi < 0, \quad \eta = 1, \tag{28}$$

$$\Psi_2 = 0 \quad \text{for} \quad -\infty < \xi < 0, \quad \eta = 0, \tag{29}$$

$$\Psi_2 = -\eta \quad \text{for} \quad \xi = -\infty, \quad 0 \leqslant \eta \leqslant 1, \tag{30}$$

$$\Psi_2 = g(\eta) \quad \text{for} \quad \xi = 0, \quad b \leqslant \eta \leqslant 1, \tag{31}$$

$$\Psi_2 = 0 \quad \text{for} \quad \xi = 0, \quad 0 < \eta \leqslant b. \tag{32}$$

The solution $\Psi$ exhibits a dividing streamline, along which velocity can be calculated, and therefore the pressure distribution can be computed. Now, if the upper region, namely the part that flows into the fictitious sink distribution introduced on $\xi = 0$, is replaced by a stagnant layer of fluid of a stable stratification, and if the static pressure produced by the stagnant layer is equal to the pressure computed before, then this is a solution to the original boundary-value problem.

The inverse method consists of the suitable choice of the sink distribution such that the velocity along the dividing streamline satisfies the conditions that $q'^2$ be monotonically increasing and $d^2(\frac{1}{2}q'^2)/d\eta^2 = 0$ along it (i.e. $q'^2$ is linear in $\eta$ along it). For, when these conditions are satisfied, a unique constant density of the stagnant wedge is determined by virtue of equation (21).

Returning to the solution of $\Psi_1$ and $\Psi_2$ and after some simple calculations, we find that

$$\Psi_1 = -\eta - \sum_{n=1}^{\infty} \frac{2}{n\pi} e^{a_n \xi} \sin n\pi\eta, \tag{33}$$

where

$$a_n^2 = n^2\pi^2 - F^{-2}. \tag{34}$$

For $\Psi_2$, it is necessary to choose the form of $g(\eta)$. A sink distribution of uniform strength from $\eta = b$ to $\eta = 1$ is assumed. The particular choice of a sink distribution is not really important here. This is analogous to the problem of Rankine bodies, where different singularity distributions can give the same solution for the external flow. Thus, the point here is to find a sink distribution that will give a $\Psi$ that satisfies the boundary conditions discussed earlier. A specific choice therefore does not restrict the class of solutions.

It is then found that

$$\Psi_2 = -\eta + \sum_{n=1}^{\infty} \left(\frac{2}{1-b}\right)\left(\frac{1}{n\pi}\right)^2 \sin n\pi b \, e^{a_n \xi} \sin n\pi\eta. \tag{35}$$

Finally

$$\Psi = -\eta - \sum_{n=1}^{\infty} \frac{2r}{n\pi} e^{a_n \xi} \sin n\pi\eta + (1-r) \sum_{n=1}^{\infty} \left(\frac{2}{1-b}\right)\left(\frac{1}{n\pi}\right)^2 \sin n\pi b \, e^{a_n \xi} \sin n\pi\eta. \tag{36}$$

The series and its differentiated series converge uniformly for all values of $\xi < 0$ and $0 \leqslant \eta \leqslant 1$. The dimensionless velocity components of the associated flow field are given by $-\partial\Psi/\partial\eta$ and $\partial\Psi/\partial\xi$. Thus, for any assumed value of $r, b, F$, the velocity along the dividing streamline can be calculated. From this a graph of $(q'^2)$ against $\eta$ is plotted to see whether a straight line is obtained, for when $(q'^2)$ is linear in $\eta$ then the dynamic boundary condition along the dividing streamline is satisfied by virtue of equation (21). The detailed calculations involved a trial-and-error process, involving variations, of $r, b$, and $F$, and were done with the aid of an IBM 7090 computer. The final choice was made on the straight line that had the smallest slope. A line of zero slope indicates $\rho_s = \rho_A$ from equation (21). Any slope away from zero contributes an error as discussed and estimated in §5 below. Hence, the line with smallest slope is chosen. Were it not for this consideration, there would be an infinite class of essentially different flows. However, with this restriction, the indeterminacy is removed. Setting $d = d_1$, we have thus obtained the solution. By virtue of the similarity of all velocity profiles, one velocity profile suffices for the whole class of solutions.

## 5. Results and discussion

The solution found is for $F = 0{\cdot}345$, $r = 0{\cdot}51$, $b = 0{\cdot}14$, and figure 2(*a*) shows that $(q'^2)$ varies linearly with $\eta$ and that it has a slope of $(-0{\cdot}9)$. Therefore from equation (21) we have

$$(\rho_s - \rho_A) = 0{\cdot}9F^2(\rho_0 - \rho_s),$$

or

$$(\rho_s - \rho_A) = 0{\cdot}1(\rho_0 - \rho_s), \tag{37}$$

which shows that $(\rho_s - \rho_A) \ll (\rho_0 - \rho_s)$. The error involved from the non-zero slope of $(q'^2)$ versus $\eta$ is consequently small, and the Froude number obtained herein, i.e. $F = 0{\cdot}345$, is indeed the unique Froude number that is being sought. Figure 2(*b*) shows the flow pattern into a sink together with the fictitious sink distribution, and figure (3) shows the flow profile for all separated flows. The flow pattern in figure (3) of the separated flow compares rather well with the photograph taken by Debler (1959) in his experiment. That the number found here is a reasonable figure of the unique Froude number can be seen by the following consideration. For $F$ slightly bigger than $\pi^{-1}$, Yih's solution (1958) shows a large eddy which is nearly horizontal, resulting in return flow to infinity. These eddies are, moreover, unstable. This indicates that a flow with discontinuity in the flow field, as given here, is relevant, the flowing part possessing a Froude number of magnitude somewhat greater than $\pi^{-1}$ (as indeed found here). The result produced with discontinuity in the flow field is then the desired solution.

Experimental values of Debler (1959) indicate that the Froude number for all separated flows lies in the neighbourhood of 0·28. The discrepancy with the number obtained here is actually superficial rather than real. This is because of the fact that in the experimental measurements the effect of viscosity tends to make the depth of the stagnant zone much smaller, so that for the same discharge

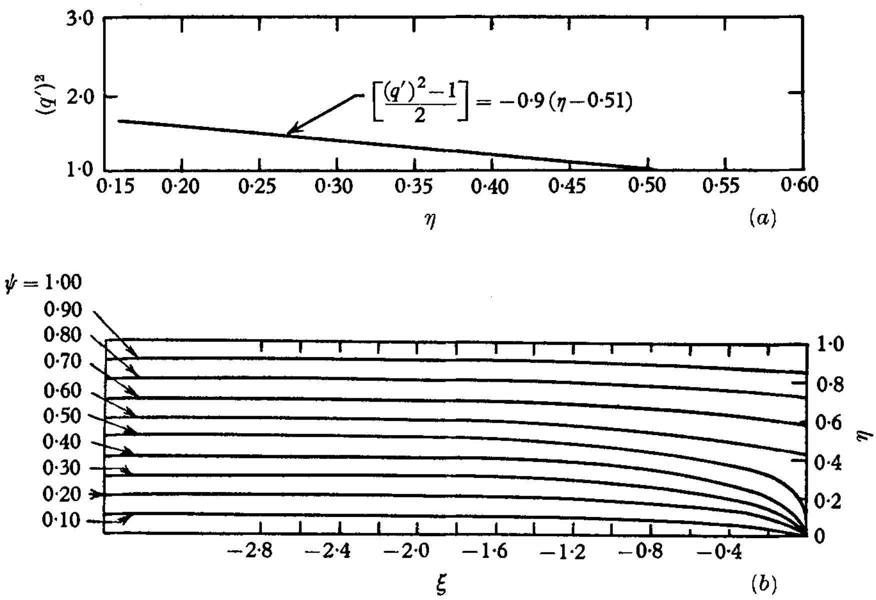


FIGURE 2. (*a*) Graph of $(q'^2)$ versus $\eta$ along the dividing streamline. (*b*) Flow pattern into a sink together with the fictitious sink distribution.

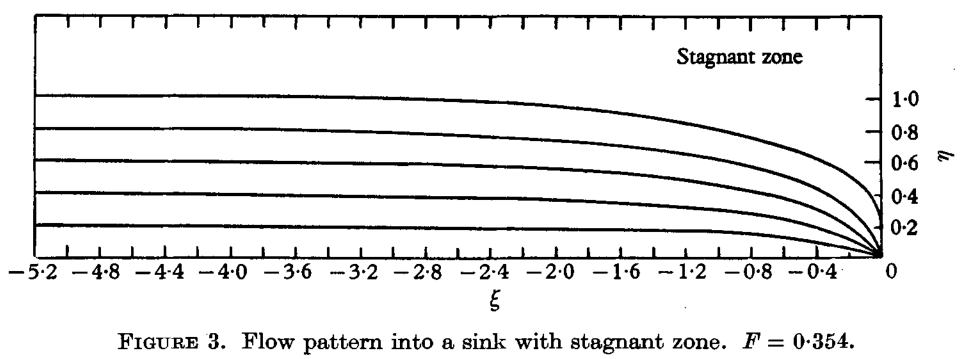


FIGURE 3. Flow pattern into a sink with stagnant zone. $F = 0{\cdot}354$.

the measured $d_1$ is bigger in the case with viscosity than if viscosity is completely absent. Also, the presence of the boundary layer at the bottom of the channel in the experimental case increases the observed depth of the flowing zone. Furthermore, the side-wall effect also tends to reduce the actual discharge compared with the theoretical discharge. Since the error in the depth of the flowing zone enters as a squared term, the experimental values when suitably corrected are in agreement with the results obtained here.

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